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TENSILE STRENGTH OF CONCRETE IN TENSION, FLEXURE, AND TORSION

by

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This Thesis is respectfully submitted to the Graduate Faculty of Lehigh University in partial fulfillment of the requirements for the degree of Master of Science.

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ACKNOWLEDGMENT

The specimens whose test results are reported in this thesis were made and tested in the Fritz Engineering Laboratory at Lehigh University. Acknowledgment is due the Laboratory for furnishing the concrete materials for the specimens; to Inge Lyse, Research Professor of Engineering Materials, for his helpful cooperation and valuable advice throughout the investigation, and to the members of the Fritz Engineering Laboratory Research Staff for assistance in the preparation and testing of specimens.

I. SYNOPSIS

Whether concrete is tested in torsion, flexure, or direct tension, the primary cause of failure in each case is the low tensile strength of the concrete. Accordingly, a definite correlation between tensile strengths obtained by the different methods of testing might be supposed to exist. However, very little data have been presented to substantiate or disprove such a hypothesis, due largely to the complexities involved in analyzing a material strained beyond the elastic limit. As a result the term, modulus of rupture (apparent tensile strength computed by formulae from the elastic theory), has come into use and has given rise to much confusion regarding the true strength of concrete in tension.

In this thesis are presented the results of torsional, flexural, tensile, and compressive tests on one-hundred-forty plain concrete specimens. The major conclusions drawn from the tests may be summarized by the following:

1. The elastic theory does not apply to plain concrete members.

2. The modulus of rupture of plain concrete members depends upon the shape of cross-section as well as upon the type of test.

3. Tensile failures of concrete obtained by testing in torsion, flexure and direct tension cannot be correlated by merely assuming that plastic redistribution of stresses (any amount) takes place in torsion and flexure members.

4. Factors which lay beyond the scope of this research influence the strength of concrete in torsion, and probably in flexure.

II. INTRODUCTION

A. Purpose and Scope - The elastic theory has long been the standard basis of analysis and design of reinforced concrete. It is only recently that engineers have begun to demand a more exact basis for design, which does not neglect the important fact that concrete flows plastically under stress. C. S. Whitney[1*] recently presented a paper on the analysis of reinforced concrete members subjected to flexural and direct stress, in which he bases his attack on the actual plastic behavior, and not the supposed elastic behavior of concrete.

To promote this trend of thought, this thesis was designed to show how the plastic nature of concrete rendered incorrect the formulae of the elastic theory, and to indicate a better procedure to follow in various cases. Since this research is basic in nature, it was decided to study only plain concrete specimens, and not complicate the issue by the addition of reinforcing steel. By testing plain concrete specimens to failure in torsion, flexure and tension, it was intended to show the differences between actual tensile strengths and the moduli of rupture (apparent tensile strengths based on elastic theory) in

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* These numbers apply to references given at the end of the thesis

flexure and torsion. The problem was extended to show the effect of shape of specimen in the torsion tests, both rectangular and cylindrical beams being studied.

As an auxiliary study, it was desired to know what effect various types of loading had upon flexural tests, or more specifically, how center-point and third-point loading compared. In the past, different investigators have used each of the two loadings in flexure studies, and it is desired to know if the results were affected by the difference in method used.

B. Historical Background. 1. Upton - In 1915 Upton[2] presented a method of determining the true ultimate tensile strengths of materials stressed beyond the elastic limit for torsional tests of cylindrical specimens and for flexural tests of rectangular specimens. He arrived at some simple formulae by assuming that the curve obtained by plotting modulus of rupture against strain of extreme fiber was horizontal at the point of failure, which is near to if not precisely the truth in many engineering materials. The results can be most easily expressed as corrections applied to the usual formulae of the elastic theory. For cylindrical specimens subjected to torsion,

$$V_m = \frac{3}{4}\frac{Tr}{J}$$

For rectangular specimens subjected to flexure,

$$S_m = \frac{2}{3}\frac{Mc}{I}$$

The derivations of these formulae are presented in the appendix.

2. In 1935 Gilkey[3] presented data from compressive, tensile, torsional, and flexural tests on plain concrete performed in connection with the Arch Dam Investigation[4] sponsored by the Engineering Foundation in 1928-29. Compression tests were made on 3 by 6-in. cylinders, tension and torsion tests on 3 by 12-in. cylinders, and flexure tests on 3 by 3 by 40-in. beams loaded at the center. To get "true" ultimate strengths in torsion and flexure, Upton's corrections for materials stressed beyond the elastic limit were applied to the results. It is well to mention here that Upton's correction for rectangular flexural members should not be applied to plain concrete beams, for it assumes that considerable plastic flow takes place on both tension and compression sides of the beam. Such is not the case, for at the low stresses at which failure occurs, the concrete in compression is still behaving elastically for all practical purposes. The adjusted torsion strengths gave good agreement with the strengths obtained in direct tension. The adjusted flexural strengths were high by about twenty-five per cent, but the discrepancy would have been smaller had the proper correction been used.

3. Gonnerman and Shuman[5], in their paper on COMPRESSION, FLEXURE, AND TENSION TESTS OF PLAIN CONCRETE, state that: " -- for concretes having compressive strengths of from 2000 to 5000 lb per sq in. -- the tensile strength ranged from about 0.5 to 0.6 of the modulus of rupture." The tensile tests were made on 6 by 18-in. cylinders and the flexural tests on 7 by 10-in. beams loaded at the third-points of a 36-in. span. This relation was true for both gravel and crushed stone concretes.

4. Morsch. In connection with a torsion investigation of reinforced concrete beams, Morsch[6] found the torsional modulus of rupture of plain concrete cylinders, 40 cm. in diameter, to be 1.65 times the strength of the concrete in direct tension. Morsch showed that this discrepancy could not be explained even by assuming the extreme case of uniform distribution of shearing stress (rectangular distribution) over the cross-section of cylinder (which is merely another way of expressing Upton's correction). See Fig. 1(c).

However, Morsch did achieve a satisfactory explanation by taking into consideration the difference between the tensile and compressive moduli of elasticity. Due to this difference, longitudinal stresses and strains are introduced into the cylinder, which cause the maximum tension at the surface to be less in magnitude than the maximum shearing stress, strengthening the cylinder. A semi-parabolic distribution of shearing stress was used in this analysis. A detailed analysis of this problem will be found in the appendix.

Morsch also tested some rectangular beams in torsion, but only two specimens were unreinforced; so the results will not be mentioned.

C. THEORETICAL CONSIDERATION

In the analysis of a cylindrical member subjected to torsion by elastic theory it is generally assumed that shearing stress at any point on a cross-section is proportional to the distance from the center of the cylinder, as in Fig. 1(a) (modulus of rupture). However, in concrete beams, plastic flow (or time-yield) takes place and the stress distribution becomes similar to that in Fig. 1(b). If it is considered that the stress-strain diagram becomes horizontal before failure, the rectangular distribution in Fig. 1(c) may be approached as a limit, though probably never achieved. This last condition is identical with the result obtained by Upton using the method outlined in the appendix.

For rectangular beams in torsion Fig. 2(a) indicates the stress distribution on a cross section and gives the formula for maximum shearing stress (at the center of the long side) according to the elastic theory. Fig. 2(b) gives the stress distribution and formulae for rectangles of three side ratios based upon the limiting case of uniform or rectangular stress distribution.

In all computations for rectangular beams in flexure, it has been assumed that strain is proportional to distance from the neutral axis (see Fig.3(a)) and that the neutral axis remains at the center of the beam. That the latter assumption is true in plain concrete beams is not definitely known, but since there are little data available on the subject, there is practically no other course open. Fig. 3(b) indicates the stress distribution according to elastic theory. If the material on both tension and compression side of the beam were to flow plastically as in a ductile material, the condition illustrated in Fig. 3(c) would be approached (Upton's method). However, at the low stresses at which concrete fails in tension, the amount of plastic flow in compression is scarcely more than a negligible quantity. Considering that the concrete in compression is still elastic while that on the tension side passes the elastic limit, gives rise to the stress distribution in Fig. 3(d) which approaches as a limit that shown in Fig. 3(e). The derivations for all coefficients shown in Fig. 1, 2, and 3 are given in the appendix.

To apply the theory developed by Morsch, it is required that the ultimate shearing strain be known, which data were not observed in this investigation. However, it is felt that the difference between moduli of elasticity in tension and compression at the ultimate tensile strength is a matter which should not be overlooked in an exact analysis of concrete in torsion.

III. TEST PROGRAM

A. General - Concrete specimens were made in ten groups with concrete strengths varying from 1900 to 4500 p.s.i. Each group consisted of six 3 by 6-in. compression cylinders, three 3 by 24-in. tension cylinders, three 3 by 30-in. torsion cylinders, three 4 by 4 by 40-in. beams (two for flexure, one for torsion), one 4 by 6 by 40-in. torsion beam, and one 4 by 8 by 40-in. torsion beam.

Ultimate strengths were obtained for all specimens. Stress-strain data were observed on six sets of compression specimens and on four sets of tension specimens. The test program for a typical test group is presented in Table I.

B. Preparation of Specimens - The concrete of the test specimens consisted of crushed limestone of three-quarter inch maximum size, a fine sand from northern New Jersey, tap water, and a standard Portland Cement produced in the Lehigh Valley. The proportions of the individual batches are given in Table II. The concrete was mixed in a two-and one-quarter cubic foot mixer of the "stirrer" type, each batch being mixed for three minutes. Two batches of approximately two cubic feet each were poured for each test group. Specimens were poured in oiled steel forms. After about two days in the forms, the specimens were stored in a moist room at 70°F and one hundred per cent humidity until they reached the test age of twenty-eight days. There were a few minor variations from this schedule, which is also presented in Table II. Specimens left in the forms for more than two days were covered with burlap and sprinkled with water at frequent intervals. Fig. 4 shows the specimens in the forms shortly before removal. A slump of four to six inches was maintained in all mixes.

During the interval between removal from the moist room and time of testing, the specimens were wrapped in wet burlap.

C. Test Methods. Compression Tests -Compression cylinders were tested in a 50,000-lb. capacity screw-driven machine, the load being applied at a speed of 0.05 in. per minute. To eliminate eccentricity the load was applied to the cylinders through a spherical bearing block. On the cylinders being tested for modulus of elasticity, strains were observed with Huggenberger tensometers with one-inch gage lengths. The steel points of the tensometers were seated directly on the concrete, the only care observed being to avoid the small irregularities that are present on the surface of concrete cylinders. Strain readings were observed at every one or two thousand pounds of load, depending on the strength of the cylinders. The number of increments at which observations were made varied from eleven to twenty. The instruments were removed at about sixty to seventy per cent of the ultimate load.

2. Tension Tests. Tension cylinders were tested in a 2000-lb. capacity hand-powered machine. The tension grips consisted of sections of three-inch steel tubing cut in half along

the longitudinal axis, welded to split end-plates, and bolted together again through small angles welded to the tubing. The load was applied to the grips through spherical bearings. The details of the grip are shown in Fig. 5. To secure uniform bearing on the cylinder, and to provide sufficient friction between the metal grip and the concrete cylinder, a quadruple thickness of ordinary paper hand towel was used as a lining. Some slipping in the grips was encountered, but it was infrequent and was always easily remedied by further tightening of the grips.

To accurately measure the strain in the tensile specimens, a twelve-inch strain gage was developed. It consisted of two circular collars twelve inches apart, each firmly attached to the cylinder by three bearing screws. The strain was measured by two 1/10,000-in. Ames dials, which were attached to one collar by long arms, and whose plungers rested upon flat plates protruding from the other collar. The two dials were situated diametrically opposite and equidistant from the cylinder. The apparatus is also illustrated in Fig. 5. It was possible to interpolate to tenths of the ten-thousandth divisions of the dials and thus read the strain to 1/120,000-in. per in. Strain readings were observed at every one hundred pounds of load, and readings were taken all the way to failure.

3. Torsion Tests. All torsion specimens were tested in a 26,000 in-lb. capacity hand-powered torsion machine. Torque was transmitted to the specimens through a head-plate and

and a pair of angles as illustrated in Fig. 6. Some strain data were observed, but they were inconsistent throughout, so the practise was abandoned.

The same grips used in tension tests were used in the torsion tests on the 3 by 30-in. cylinders. A double thickness of paper lining was used, and no slipping of the grips was encountered.

For rectangular torsion specimens the angles bolted to the head plate of the torsion machine served as grips, and pieces of beaver-board 3/16-in. thick were used for liners to distribute the pressure. Fig. 7 shows a specimen in the torsion machine ready for testing, and Fig. 6 and 8 show specimens already tested.

4. Flexural Tests. Flexural tests were made in the 2000-lb. capacity hand-powered testing machine. In each test group one specimen was loaded at the center and one at the third-points. Ultimate strengths only were observed.

IV. TEST DATA

A. Failure of Specimens - More than three-fourths of the tension specimens broke within the middle third of the length, indicating that the pressure of the grips had little effect upon the strengths obtained. Typical tension failures are shown in Fig. 9 and 10. Every precaution was taken to eliminate eccentricity from the tension testing apparatus, but it seems probable that these efforts were not wholly successful.

In some of the few modulus of elasticity tests that were performed, measurable eccentricity was present.

Almost all of the circular torsion specimens broke near the middle of the specimen with the usual 45-degree spiral form of failure, similar to that obtained by twisting a piece of ordinary blackboard chalk. See Fig. 11 and 12. Only about two-thirds of the rectangular torsion specimens broke away from the grips. Of these, the square specimens broke most consistently near the center. Nevertheless, the strengths of those specimens that broke at or near the grip were not unduly lower than the strengths of the other specimens. Typical rectangular torsion failures are shown in Fig. 13.

B. Ultimate Strengths - The ultimate strength results are presented in Table III in the form of unit stresses, actual ultimate stresses for tension and compression tests, and moduli of rupture for torsion and flexure tests. For computing moduli of rupture, the usual formulae of the elastic theory, given in Fig. 1, 2, and 3, were used. Too much weight should not be given to individual results in the lower five lines of the table, for most of these values represent only one specimen. As indicative of general trends, however, each group as a whole must be given due consideration.

In Fig. 15 and 16, tensile strengths and moduli of rupture have been plotted against compressive strengths, and straight lines have been fitted to the points representing each individual type of test. These diagrams show the relationship between tensile strength and the various moduli of rupture, as well as between each of those items and the compressive strength. It is well to note that all rectangular torsion strengths have been plotted as one group. Since the modulus of rupture was very nearly the same for each of the three rectangles for any concrete strength, the results have been plotted for the sake of clarity as the averages of the three specimens. In Fig. 17 the moduli of rupture for the 1:1, 1:1-1/2, and the 1:2 rectangular torsion specimens have been plotted individually, showing what slight variations did exist.

C. Ratios Between Strength Properties - In Table IV are expressed the relationship between the strength obtained in direct tension and the other strength values by giving the ratio $\frac{\text{Tension Strength}}{\text{Ultimate Strength or Modulus of Rupture}}$ for each test.

The tension ultimate strengths range from 6 to 10 per cent of the compression, 45 to 65 per cent of the circular torsion, 30 to 45 per cent of the rectangular torsion, and 35 to 50 per cent of the flexure. From these ratios it may also be derived that for the rectangular torsion specimens, the modulus of rupture increases slightly with the increase in side-ratio, as was also indicated by Fig. 17. Flexure specimens loaded at the center consistently gave a modulus of rupture about ten per cent higher than those loaded at the third-points.

D. <u>Rectangular Stress Distribution</u> - Some writers have advanced the opinion that the chief reason for the discrepancies observed between tensile strengths and moduli of rupture is the type of stress distribution. As has been previously stated in this thesis, the limiting condition, assuming considerable plastic flow to take place, is the rectangular distribution of stress, - over the whole section in torsion, and over the tension half of the beam in flexure. In Fig. 18 are plotted the values of Table III, corrected to give "true" ultimate tensile stresses in accordance with the assumption of rectangular stress distribution. To avoid confusion, the points representing two of the rectangular torsion tests and the center loaded flexure tests have been omitted, and the same straight line has been made to fit both circular torsion and third-point flexure data. Strengths for the omitted tests would in general fall between the lines representing the circular torsion and the square torsion strengths. From this diagram alone it may be concluded that other factors than the stress distribution are influencing the ultimate strength in flexure and torsion.

E. <u>Modulus of Elasticity in Tension and Compression</u> - Several types of testing apparatus for modulus of elasticity observations on tension and compression cylinders were tried out before the final set-ups were selected. The Huggenberger extensometers seated directly on the concrete of the compression specimens gave excellent results. The tensile modulus apparatus had to be used with great care, for the total measurable extension was quite small and extreme accuracy was needed. The results, however, were most gratifying.

The initial modulus of elasticity of concrete of a given strength was found to be very nearly the same for both tension and compression, that for the former being slightly higher. Initial moduli for tension and compression are plotted against compressive strengths in Fig. 19. Also in Fig.19 are plotted tangent moduli for concrete in tension at 75 per cent of the ultimate tensile strength. At such a stress the compression modulus is for all practical purposes, still the initial modulus. Typical stress-strain curves for tension and compression are shown in Fig. 20.

VII. DISCUSSION OF TEST RESULTS

A. General - In general, the test results were not as uniform as might have been expected. In a few instances, batches poured on the same day, supposedly with identical ingredients, gave decidedly different compressive strengths. Three compression cylinders were poured from each of the two batches that made up a pouring. In only one or two instances, however, was this discrepancy reflected in the strengths of the other test specimens, giving rise to the opinion that the error lay chiefly in pouring one or both of the sets of compressive cylinders. Variation in the order of pouring, and permitting the mix to dry out somewhat before pouring the small cylinders, may have had some influence.

Another possible cause for lack of uniformity in results was the variation in length of time of testing of different specimens, with the resultant drying out. When modulus of elasticity data were being observed, considerably more time was needed to test specimens than otherwise. Such drying out would tend to strengthen compression specimens and weaken tension specimens.

Comparison of the results of this thesis and those of similar researches gives indication that the direct tension values of the thesis results are generally low. Table V showing various strength ratios (flexure and torsion values are m moduli of rupture, the latter for circular specimens), gives evidence to support this contention. All thesis ratios, in which the tension value is in the numerator, are low.

Several possible causes, including drying out of specimens and eccentricity in the tension loading rig, have already been advanced. However, that either or both of these influences could be responsible for all of the discrepancy apparent in Table V is not probable. Much of these differences in the opinion of the author, must be charged to differences in materials used, or in other words, to natural causes.

B. <u>Modulus of Rupture</u> - While it has long been known that concrete does not behave elastically, the tendency to use the elastic theory in the analysis and design of concrete members has persisted. The results of this investigation show how greatly in error the formulae of the elastic theory can be when applied to plain concrete members. For engineering purposes it is well to have such tools as the modulus of rupture, but only if it is clearly understood just what it represents and approximately what its numerical value is for the particular case to which it is to be applied. The results of this thesis show that the modulus of rupture is decidedly different for flexure and torsion, and for torsion specimens of different shapes. The modulus of rupture for the rectangular torsion specimens tested in this research was approximately 3/2 of that for the circular torsion specimens. However, in view of the minor disagreement in results between this thesis and other similar researches, it is recommended that more emphasis be placed on the qualitative rather than the quantitative nature of the results.

Of special interest is the difference found between the modulus of rupture of circular and rectangular torsion members. Previously, the modulus of rupture of concrete cylinders in torsion has been occasionally used as a measure of the torsional strength of concrete for application to beams, dams, etc. The results obtained in this thesis indicate that the modulus of rupture of a cylinder is not a true measure of the torsional strength of a rectangular member by about fifty per cent. For application to rectangular members, the modulus of rupture of a similarly shaped test specimen should be used, if at all. However, for all practical purposes the modulus of rupture of a square section can be used as a measure of the torsional strength of the concrete in rectangular beams of side-ratios up to 1:2.

C. <u>Rectangular Stress Distribution</u> - The results obtained in this thesis indicate that the assumption of rectangular stress distribution will not produce agreement between the tensile strengths obtained by torsion, flexure, and tension testing. Since this assumption can be applied quite satisfactorily to tests by Gonnerman and Shuman and by Gilkey on circular torsion and rectangular flexure tests, the above statement is based chiefly on the relation of the rectangular torsion test results to the results of the other tests. Fig. 19 shows that according to the assumption of rectangular stress distribution, the circular torsion and the flexure strengths are approximately equal, but that the rectangular torsion strengths are still considerably higher. Unless other factors were present, such a condition could not exist.

The fact that the moduli of rupture for the three rectangular torsion shapes were very nearly equal (within seven per cent) gives evidence that the stress distribution is not rectangular for those specimens. For rectangular distribution to hold, the moduli of rupture would have to be in the ratio of 0.527; 0.606; 0.695 (Fig. 2(b)) for the rectangles of side-ratios, 1:2, 1:1-1/2, and 1:1, respectively. This would mean that the redistribution of stress had not reached the rectangular stage, which furthers the contention that other factors than stress distribution affect the strength of concrete in torsion.

However, an interesting point is raised in this connection by Upton's analysis of torsion and flexure for materials stressed beyond the elastic limit. In the derivation, it is proven that if the curve of apparent stress (computed by elastic theory) versus deformation becomes horizontal before failure, the true stresses in torsion are equal to $\frac{3\,Mr}{4\,J}$ and in flexure $\frac{3\,Mr}{4\,J}$. These identical results are obtained by assuming rectangular stress distribution for similar specimens. These results apply only to circular torsion specimens and rectangular flexure specimens, respectively. Thus, if the above-mentioned curve were known, a prediction as to the type of distribution and even the true ultimate stress could be made. Unfortunately, no strain data were observed for the torsion or flexure tests for this thesis.

D. Flexure Loading - The difference in results of flexure tests with center and third-point loadings is in accordance with previous findings. Talbot found in tests on reinforced concrete beams that center loading tests gave results higher than those computed by the usual beam formulas, while third-point and similar loadings gave good agreement. Two factors probably contribute to the apparent increase in strength observed in center loaded beams. The maximum moment in this case occurs only at the center of the beam, which practically forces the failure to occur at that point. When there is a considerable distance subjected to maximum moment as with third-point loading, there is usually some section slightly weaker than the rest, which will cause failure to occur somewhat earlier than in the first case mentioned. In addition, there is probably some distribution of load under each concentrated load due to the fact that the load is applied at the top of the beam, which would reduce the maximum moment slightly in the case of the center loading, but not in third-point loading.

E. Modulus of Elasticity - The agreement between initial moduli of elasticity for tension and compression of concrete is in accordance with previous findings in this country. The moduli effective at failure, however, are quite different, as indicated by Fig. 19. The modulus in compression at that point is still practically the initial modulus, while the modulus in tension is much less. It is only logical, therefore, that torsion specimens should deform in a slightly different manner near the point of failure than at lower loads. The verification of Morsch's theory, which takes consideration of this difference in modulus of elasticity, would make an interesting subject for future research on this subject.

F. Theories of Failure - Heretofore in the discussion of this thesis, it has been tacitly assumed that failure of concrete specimens tested in torsion, flexure, and direct tension will occur when the tensile stress reaches a certain maximum value. This is in accordance with the maximum stress theory of failure, which is notorious for the lack of experimental data upholding it. The question now arises - why should the tensile strengths obtained by torsion, flexure, and tension testing agree? Perhaps the material fails when the tensile strain reaches a maximum value in accordance with the maximum strain theory of failure, which does apply fairly accurately to some brittle materials. However, past experiments have shown that the maximum strains in flexure and torsion are considerably greater than that in direct tension. The actual determination of the cause of failure in concrete would necessitate a much longer and much more comprehensive investigation than that undertaken for this thesis. But until something more definite is known about the cause of failure, it is improbable that the questions raised in this thesis will ever be answered.

VIII. SUMMARY AND CONCLUSIONS

Based upon the test results presented in this thesis and with due consideration for previous work in this field, the following conclusions are made:

1. The elastic theory is inadequate for analyzing plain concrete members loaded to failure.

2. Rectangular stress distribution does not provide a wholly satisfactory solution for the differences in tensile strengths obtained by testing concrete in torsion, flexure, and direct tension. However, it is much more accurate than the assumption of linear stress distribution, as in the elastic theory.

3. The modulus of rupture in torsion is a function of the shape of cross-section as well as of the material. For rectangular concrete members of side-ratios up to 2.0, the modulus of rupture is approximately 3/2 that of cylindrical members. The modulus of rupture for rectangular torsion specimens increases slightly with increases in side-ratio.

4. Rectangular plain concrete members subjected to flexure will apparently carry about ten per cent more moment when loaded at the center than when loaded at the third-points.

5. The initial modulus of elasticity of concrete is very nearly the same in both tension and compression, the tests reported in this paper indicating that it is slightly higher for the former. However, at loads near the ultimate strength in tension, the modulus of elasticity in tension is considerably lower than that in compression. In torsion members, such a difference will affect the manner of deformation and possibly the ultimate strength.

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VITA

Winston Edward Black was born on October 26, 1915, in Chicago, Illinois, the son of Charles Edward and Cecelia DeEtte Glass Black. Primary and secondary schools attended were the H. S. Belding School and the Carl Schurz High School, respectively, both Chicago public schools. After four years at the University of Illinois, he received the degree of Bachelor of Science in Civil Engineering in June 1936. From 1936 until the present time, he has been taking graduate work in Civil Engineering as a Research Fellow at Lehigh University. All graduate courses have been studied under the following three men:

Hale Sutherland, Head of the Department of Civil Engineering,

Inge Lyse, Research Professor of Engineering Materials,

J. B. Reynolds, Professor of Mathematics and Theoretical Mechanics.

TABLE I - TEST PROGRAM FOR TYPICAL TEST GROUP

Type of Test	Dimensions of Specimen In inches	Number of Specimens	Data to be Observed
Compression	3 x 6 cyl.	6	Ultimate Strength Mod.of Elasticity*
Tension	3 x 24 cyl.	3	Ultimate Strength Mod.of Elasticity°
Torsion	3 x 30 cyl.	3	Ultimate Strength
Torsion	4 x 4 x 40	1	Ultimate Strength
Torsion	4 x 6 x 40	1	Ultimate Strength
Torsion	4 x 8 x 40	1	Ultimate Strength
Flexure (center ldg.)	4 x 4 x 40	1	Ultimate Strength
Flexure (1/3 pt.ldg.)	4 x 4 x 40	1	Ultimate Strength

* Observed in six test groups

° Observed in four test groups

TABLE II - MATERIALS AND MANNER OF CURING

Test Group	c/w Ratio (Net)	Materials — lb. Cement	Water (Net)*	Sand	Stone	Time in Form days	Age° at Test	Compressive Strength
A	1.50	34.5	23.0	79	158	1	28	3938
B	1.20	27.6	23.0	80	160	2	28	2412
C	1.10	22.2	20.0	85	168	2	28	1908
D	1.75	40.2	23.0	77	154	2	28	4420
E	1.27	29.2	23.0	80	160	2	29	3315
F	1.50	34.5	23.0	79	158	3	28	4250
G	1.75	40.2	23.0	77	154	2	28	4380
H	1.38	31.7	23.0	79	159	3	28	2800
I	1.63	37.5	23.0	78	156	2	28	4440
J	1.15	26.5	23.0	81	162	4	28	2615

* One per cent of weight of aggregate added for absorption

° Specimens kept in moist room from removal from forms until day of test

TABLE III - ULTIMATE STRESSES AND MODULI OF RUPTURE

in pounds per square inch

Type of Test		A	B	C	D	E	F	G	H	I	J
Compression*		3930	2461	1908	4420	3315	4250	4380	2800	4440	2615
Tension°		501	231	184	346	204	272	318	215	283	202
Torsion	Circular°	501	345	282	544	454	596	626	463	561	428
	Square+	865	524	450	801	599	733	833	736	760	606
	1:1/2+ Rectangular	860	700	444	---	540	661	795	667	935	669
	1:2+ Rectangular	796	634	428	828	723	841	935	814	871	709
Flexure	Center+ Loading	---	642	463	871	673	701	701	634	785	659
	1/3 Point+ Loading	721	515	376	750	646	604	657	540	748	545

* Average of six specimens

° Average of three specimens

+ One specimen

TABLE IV - RATIOS OF TENSION STRENGTHS TO COMPRESSION STRENGTH AND TO MODULI OF RUPTURE

Type of Test		A	B	C	D	E	F	G	H	I	J	Ave.
Compression		.076	.094	.096	.078	.061	.064	.072	.077	.064	.077	.075
Torsion	Circular	.605	.671	.654	.638	.451	.456	.508	.469	.505	.472	.529
Torsion	Square	.348	.441	.408	.431	.340	.370	.382	.292	.373	.333	.367
Torsion	1:1-1/2 Rectangular	.350	.330	.415	--	.377	.412	.400	.323	.303	.302	.349
Torsion	1:2 Rectangular	.377	.365	.431	.419	.282	.324	.340	.265	.324	.285	..333
Flexure	Center Loading	--	.360	.397	.390	.303	.388	.454	.339	.361	.311	.362
Flexure	1/3 Point Loading	.417	.449	.490	.461	.316	.450	.485	.398	.380	.371	.415

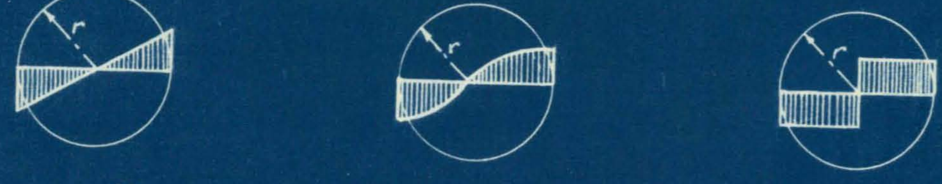
TABLE V - COMPARISON OF COMPRESSIVE, TENSILE, FLEXURAL, AND TORSIONAL PROPERTIES OF CONCRETE FOUND BY DIFFERENT INVESTIGATIONS

Ratio	Correction for Rectangular Distribution	Black Thesis	Connerman & Shuman	Gilkey	Morsch
Tension / Compression	---	0.06 to 0.10	0.08 to 0.10	0.07 to 0.11	--
Tension / Flexure	0.572	0.35 to 0.50	0.50 to 0.60	0.40* to 0.60	--
Tension+ / Torsion	0.750	0.45 to 0.65	--	0.60 to 0.80	0.61
Flexure / Compression	---	0.14 to 0.21	0.13 to 0.19	0.13° to 0.23	--
Torsion+ / Flexure	0.762	0.67 to 0.95	--	0.60* to 0.77	--

* Ratio probably somewhat low due to center loading in flexure

° Ratio probably somewhat high due to center loading in flexure

\+ Tests on circular specimens

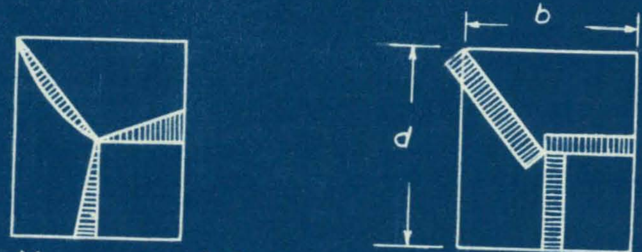


$V_m = \frac{Mr}{J}$
Modulus of Rupture
(a)

$V_m = \frac{5}{6}\frac{Mr}{J}$
2nd Degree Parabola
(b)

$V_m = \frac{3}{4}\frac{Mr}{J}$
(c)

STRESS DISTRIBUTION OF CIRCULAR SECTION IN TORSION
FIG. 1



$d = b, \quad V_m = 0.695K\frac{M}{b^3}$

$d = \frac{3}{2}b, \quad V_m = 0.606K\frac{M}{b^2d}$

$d = 2b, \quad V_m = 0.527K\frac{M}{b^2d}$

$V_m = K\frac{M}{b^2d}, \quad K = \left(3 + \frac{2.6}{0.45 + d/b}\right)$
Modulus of Rupture
(a)

(b)

STRESS DISTRIBUTION OF RECTANGULAR SECTION IN TORSION
FIG. 2

STRAIN
(a)

$f = \frac{Mc}{I}$
Modulus of Rupture
(b)

$f = \frac{2}{3}\frac{Mc}{I}$
(c)

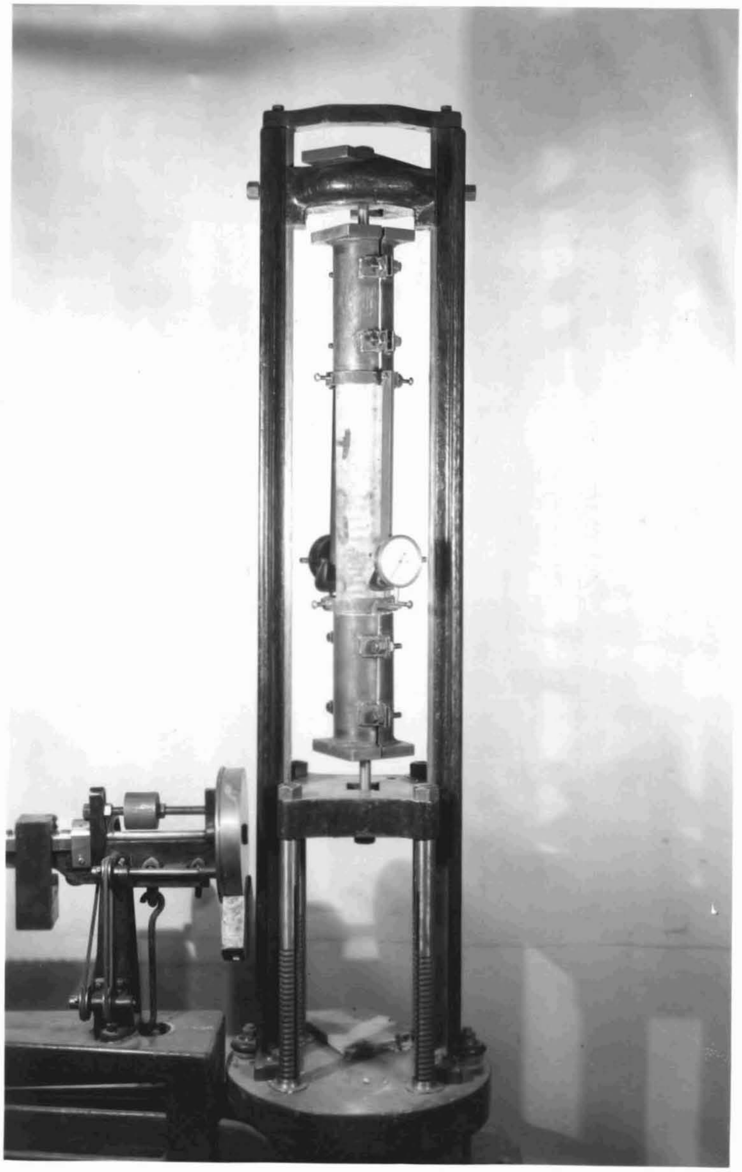
$f_t = 0.775\frac{Mc}{I}$
(2nd Degree Parabola)
(d)

$f_t = 0.572\frac{Mc}{I}$
(e)

STRESS DISTRIBUTION OF RECTANGULAR SECTION IN FLEXURE
FIG. 3

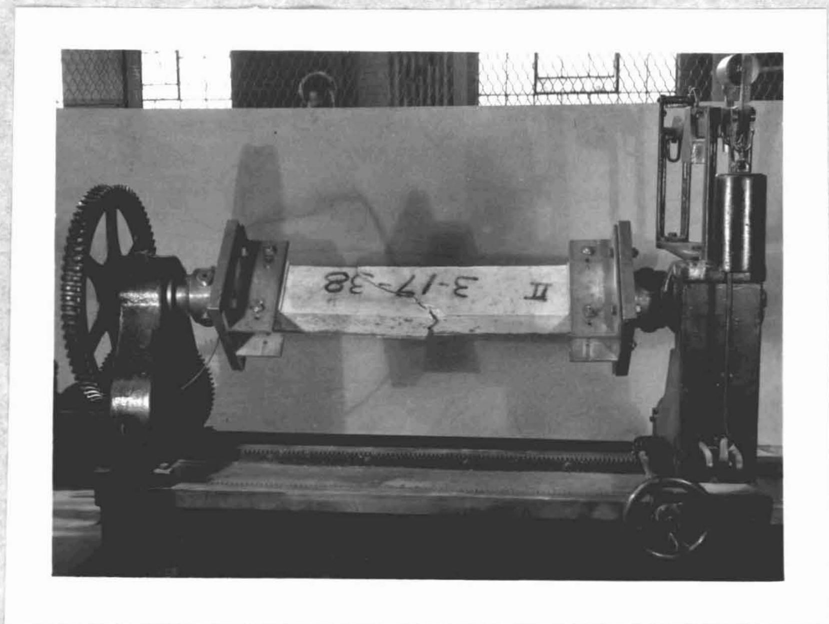
Test Specimens in Forms

Fig. 4



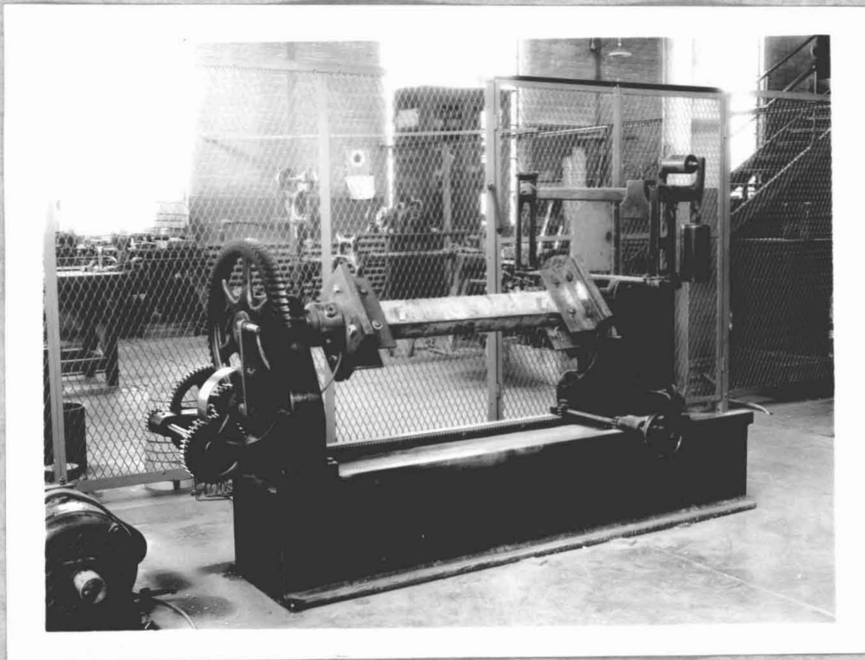
Tension Testing Apparatus

Fig. 5



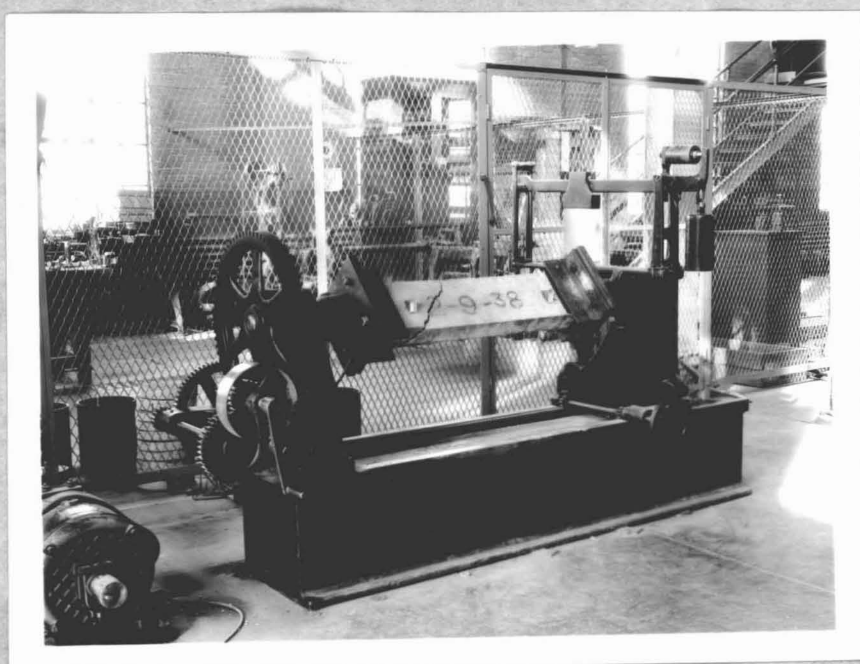
4 by 6 by 40-in. Torsion Specimen

Fig. 6



4 by 4 by 40-in. Torsion Specimen.

Fig. 7



4 by 8 by 40-in. Torsion Specimen

Fig. 8

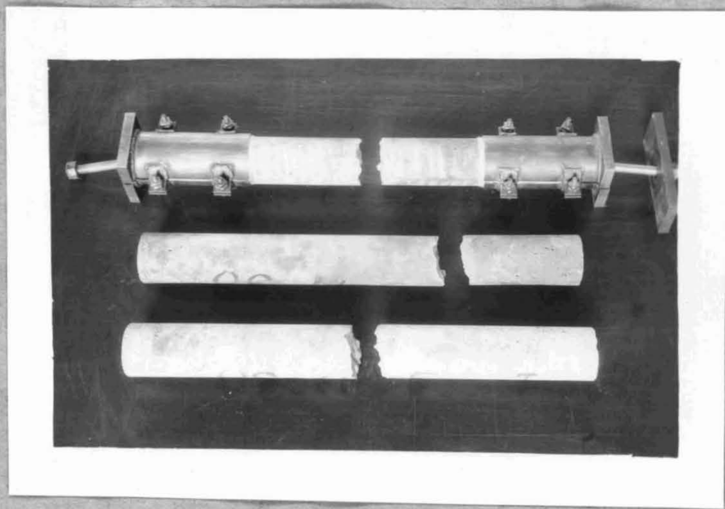


Fig. 9

Fig. 10

Typical Tension Failures

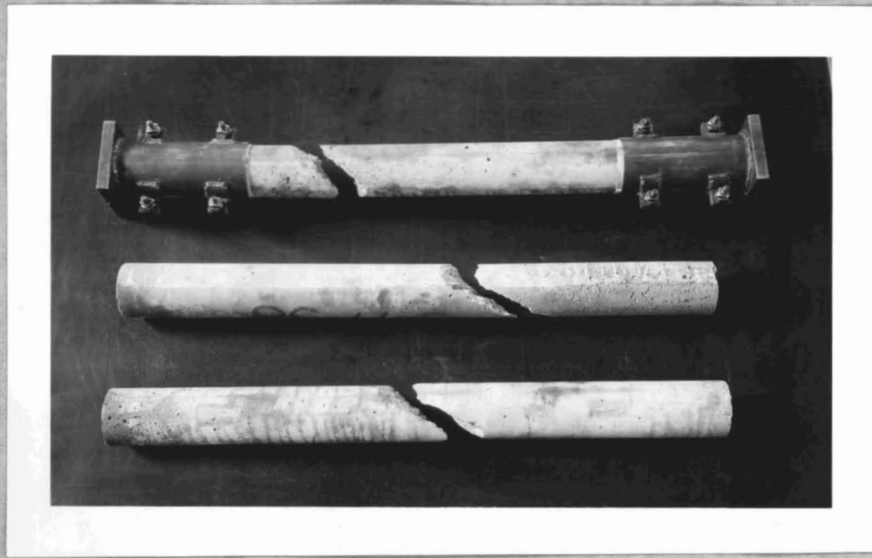


Fig. 11

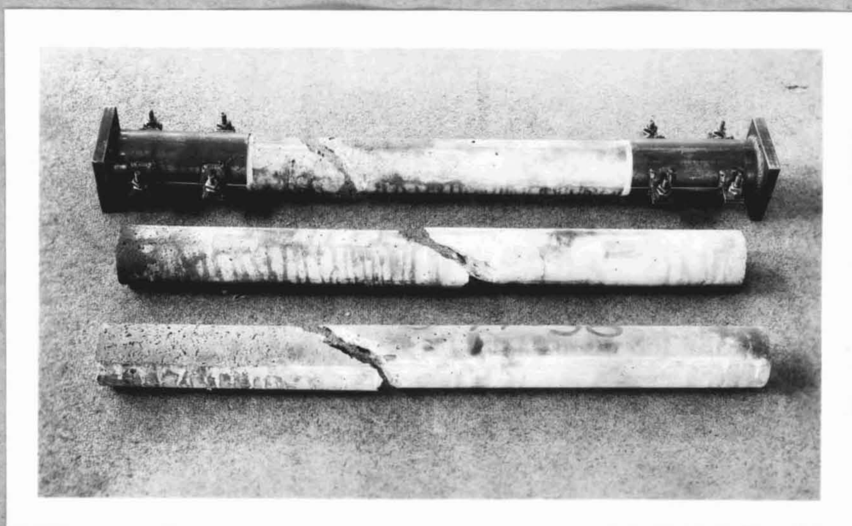
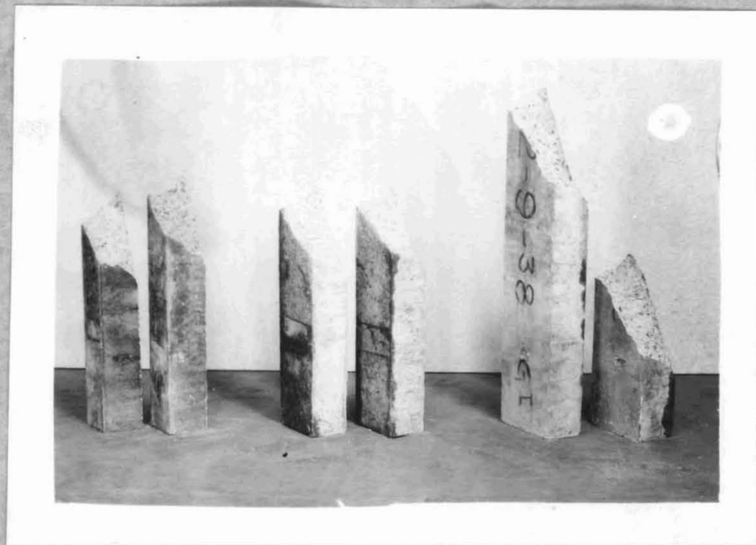


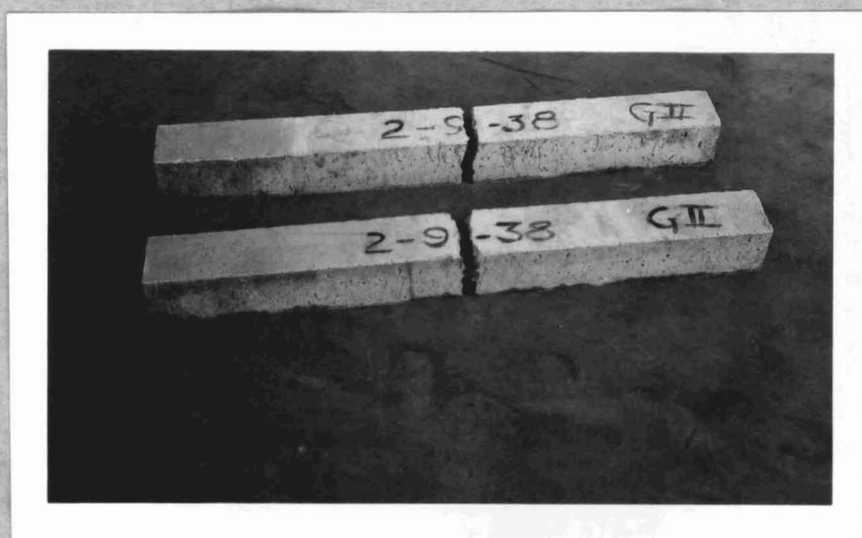
Fig. 12

Typical Torsion Failures of Cylindrical Specimens



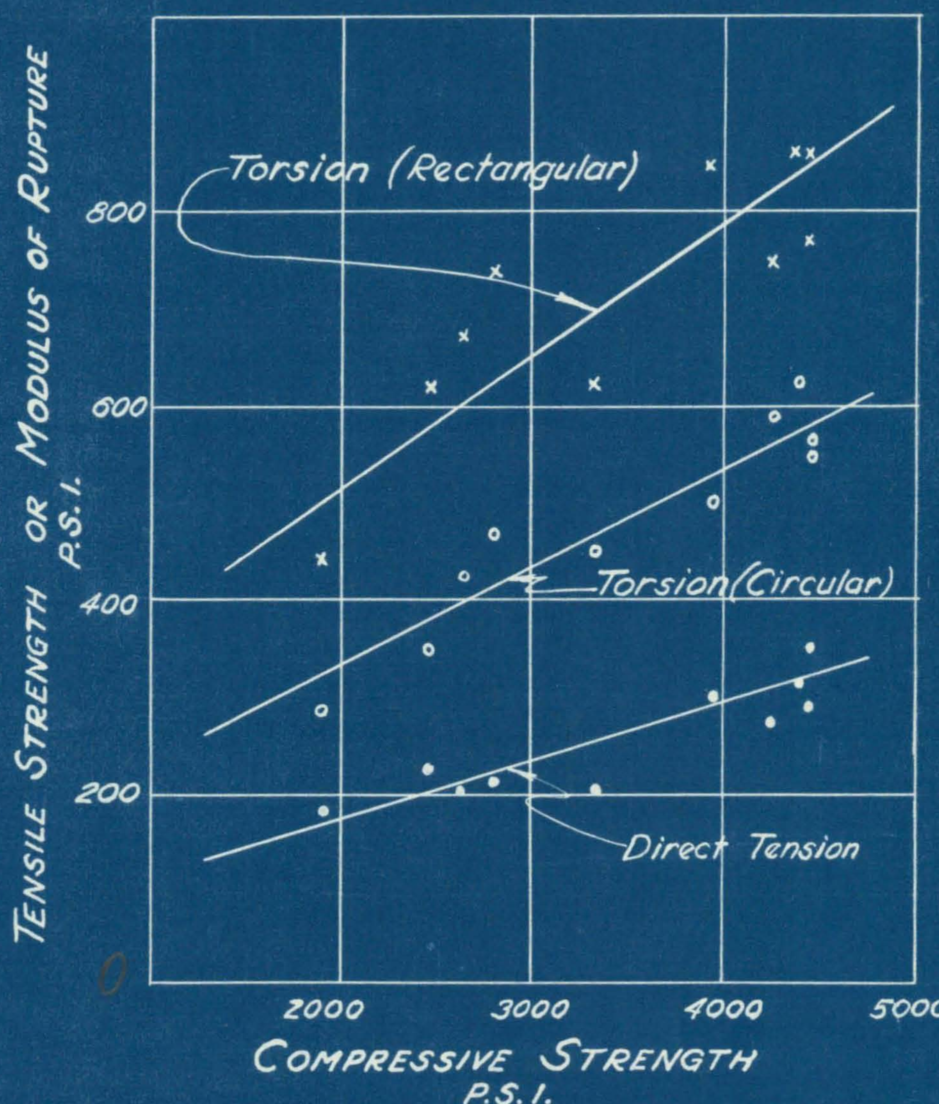
Typical Torsion Failures of Rectangular Specimens

Fig. 13

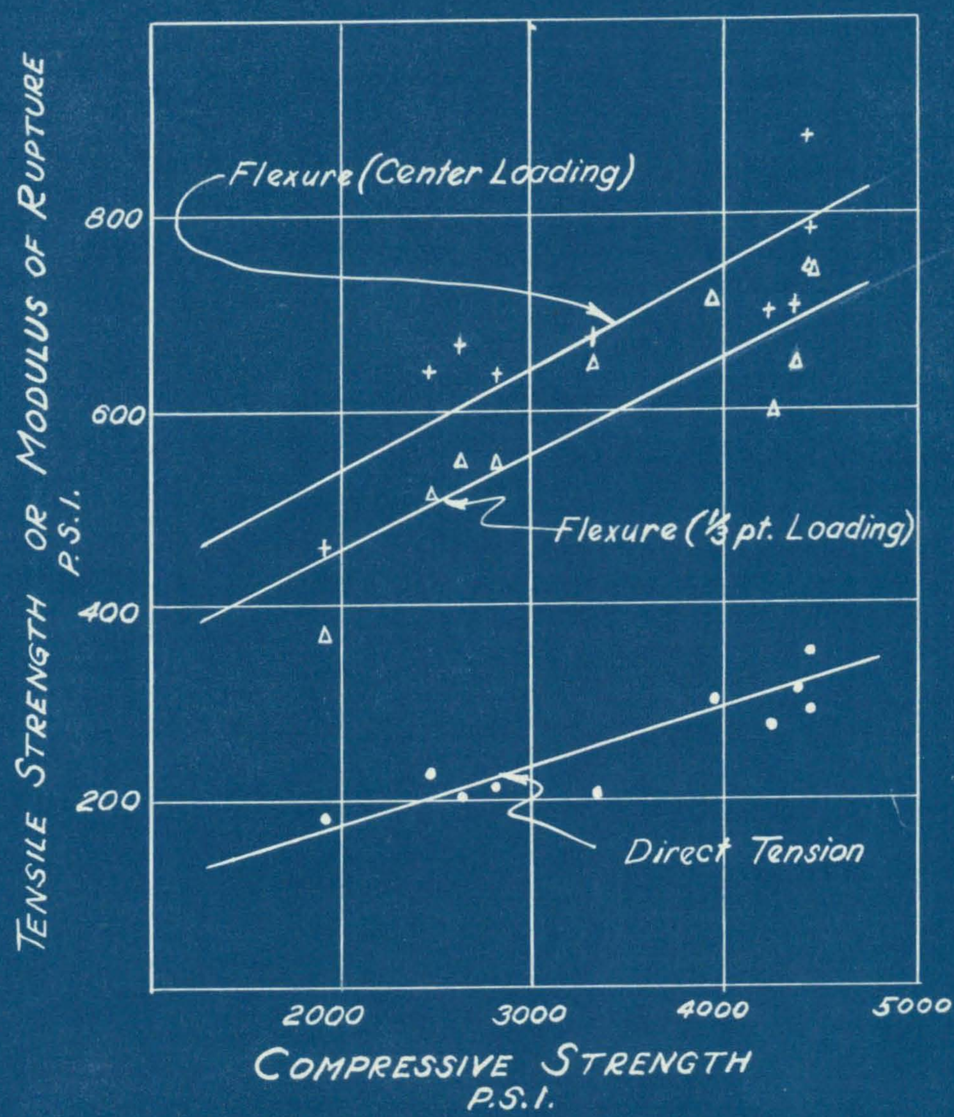


Typical Flexure Failures

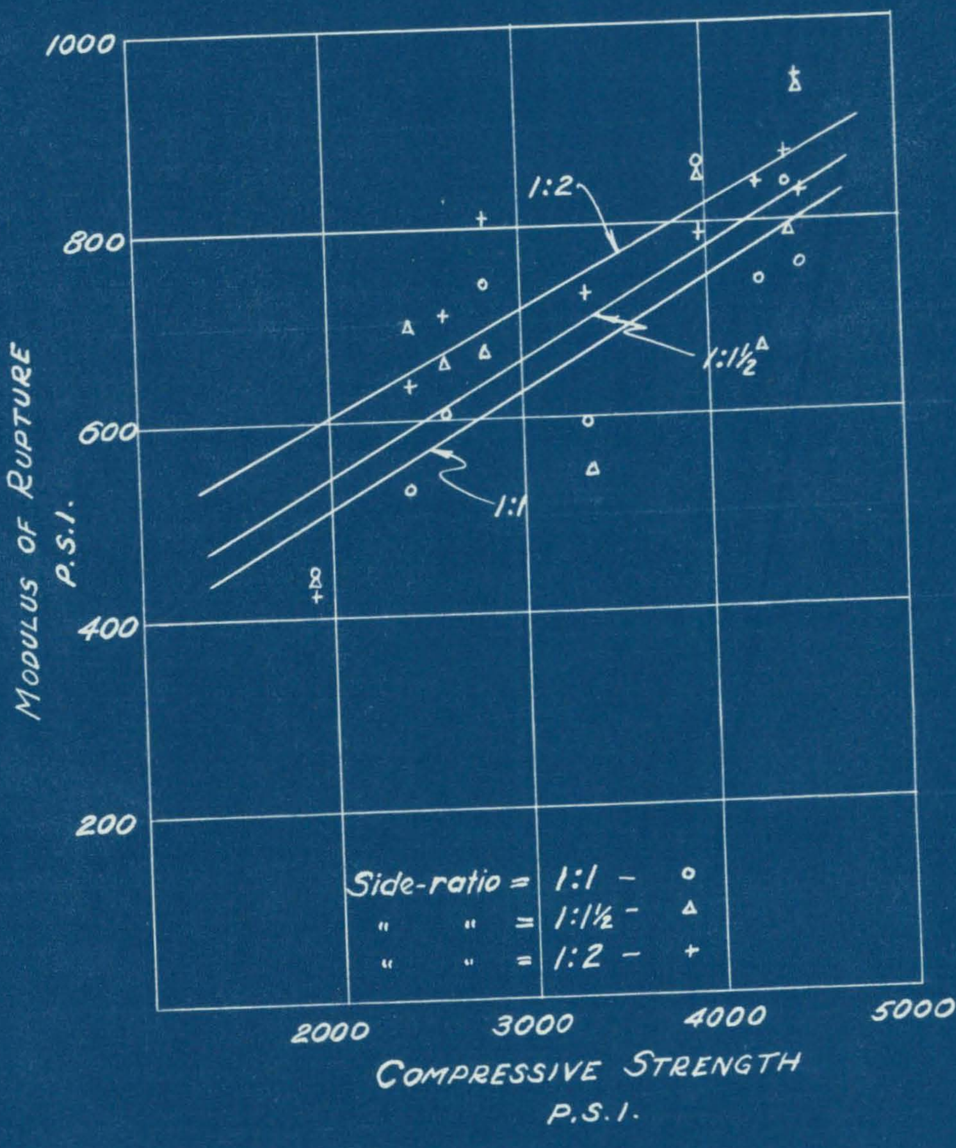
Fig. 14



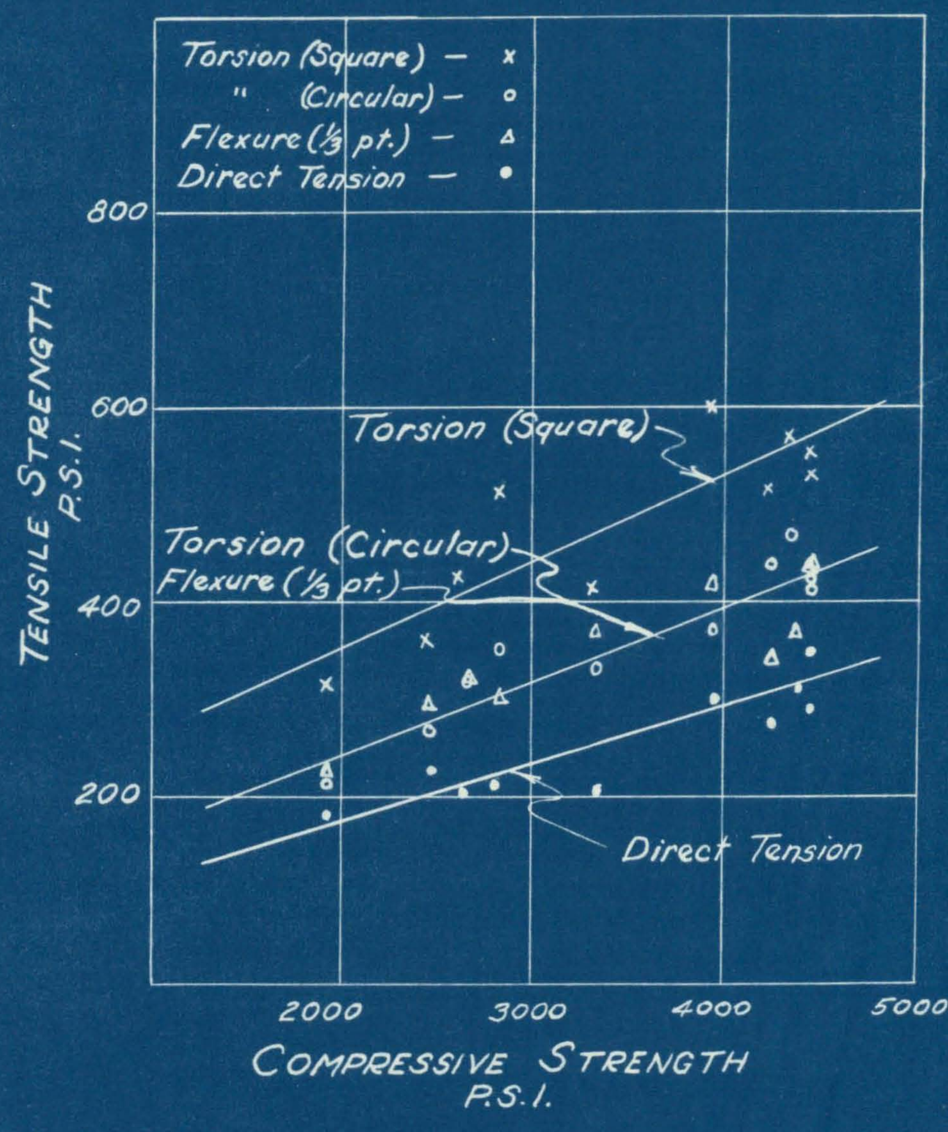
Modulus of Rupture
in Torsion
Fig. 15



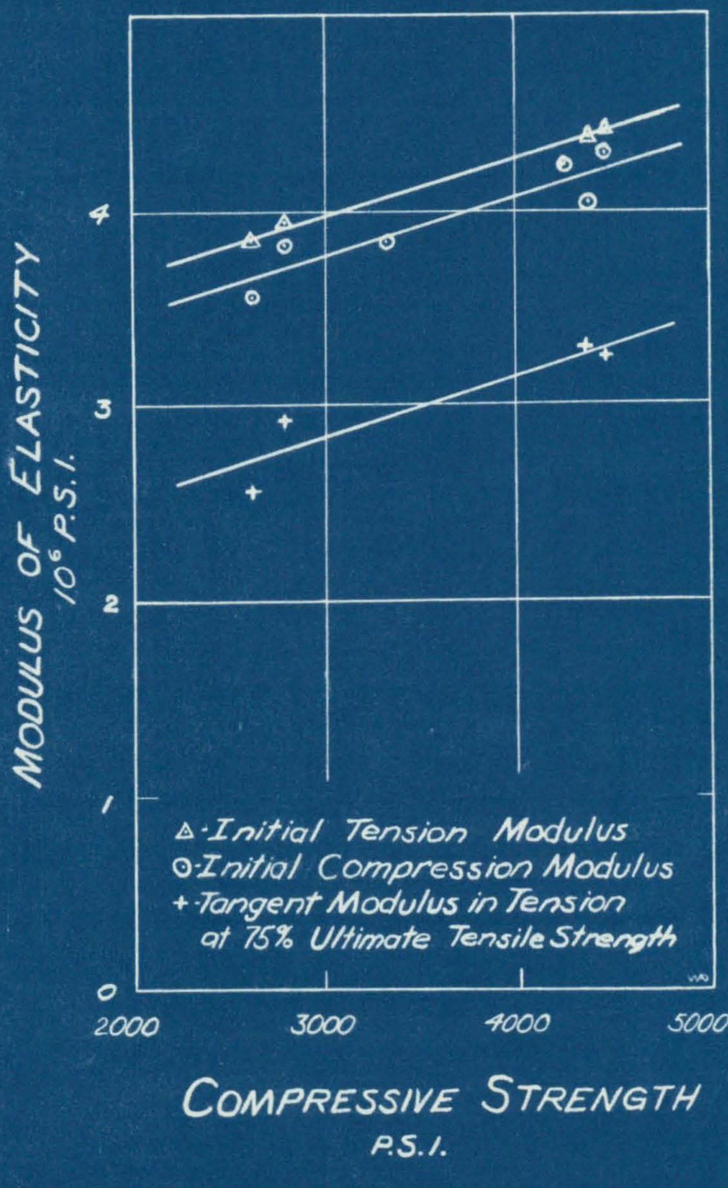
Modulus of Rupture
in Flexure
Fig. 16



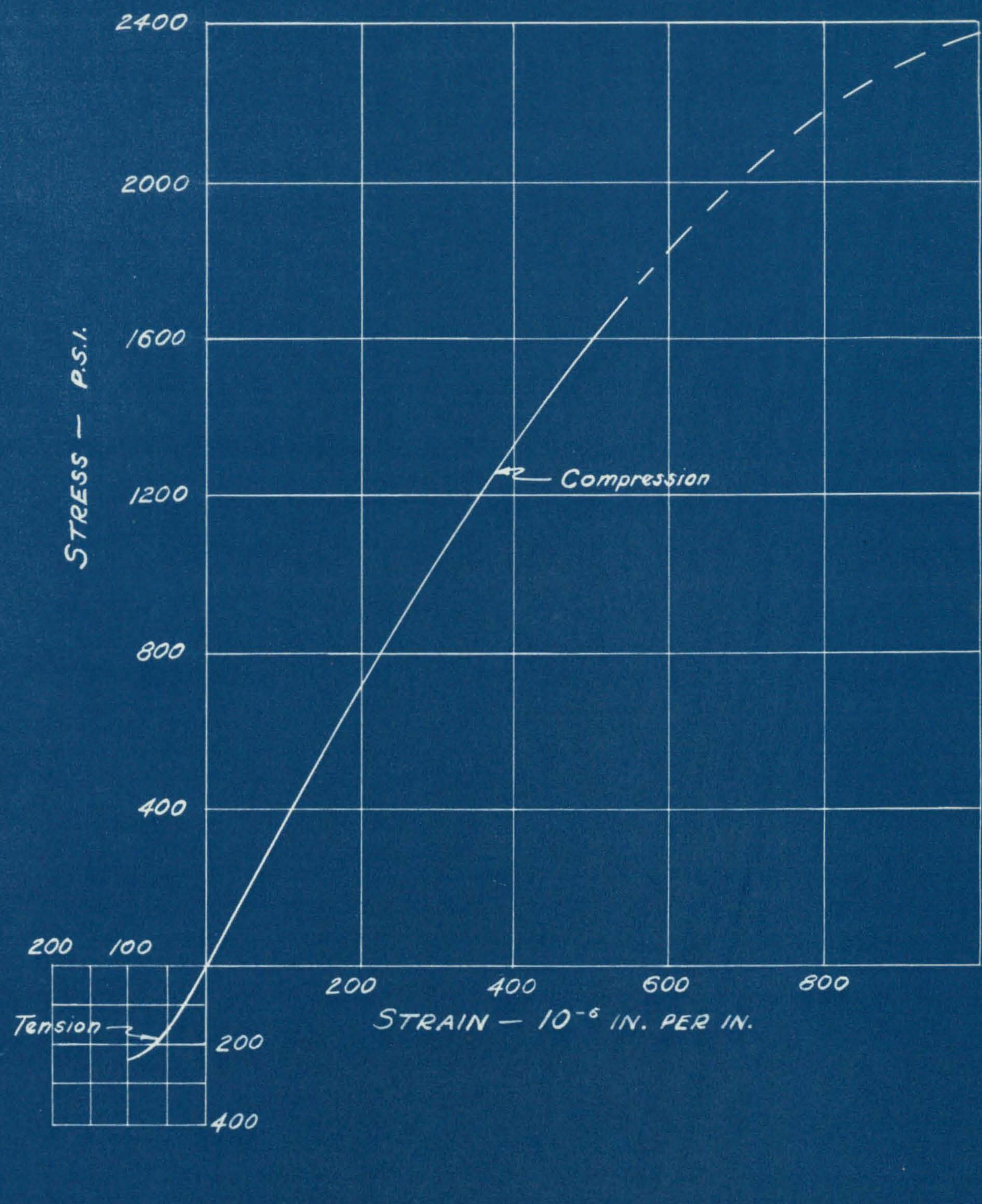
Modulus of Rupture in Torsion
of Rectangular Beams
Fig. 17



Ultimate Strengths based on
Rectangular Stress Distribution
Fig. 18



MODULUS OF ELASTICITY
IN TENSION AND COMPRESSION
FIG. 19



STRESS-STRAIN CURVES
FOR TENSION AND COMPRESSION
FIG. 20

APPENDIX I

DETERMINATION OF TRUE STRESSES IN TORSION AND FLEXURE BEYOND THE ELASTIC LIMIT AS DEVELOPED BY UPTON:

Torsion - Circular Shaft.

The following notation will be used:

M = twisting moment) measureable quantities
α = angle of twist)

l = length of specimen.

r = radius, ρ = radial distance to any fiber.

q = shearing stress, q_s = shearing stress at outer surface, q_{ns} = nominal shearing stress at outer surface by formula, $q = \frac{M\,r}{J}$

J = polar moment of inertia.

$\tan d_s$ = shearing strain at outer surface

$\tan d$ = shearing strain

From test the relation of q_{ns} to $\tan d_s$ can be measured up to failure. The relation of q to $\tan d$ cannot be measured past the elastic limit. But since the relation of q to $\tan d$ determines the form of the curve of q_{ns} against $\tan d_s$, and indirect solution is possible.

Let the relation of q to $\tan d$ be the general function $q = \psi(\tan d)$. Then q_s, the real shear stress in the surface, is $\psi(\tan d_s)$, and $\tan d_s$ can be truly found at all stages of the loading r, l, and α. As before, take as a unit for mathematical analysis a ring of material in the cross-section, of width $d\rho$, at radius ρ. Hence

$$dM = \rho\cdot q\cdot 2\pi\cdot d\rho = 2\pi q\rho^2 d\rho$$

to put ρ in terms of $\tan d$, we put first

$$\tan d = \frac{\alpha}{l}\rho \quad \text{and} \quad \tan d_s = \frac{\alpha}{l}r$$

hence,

$$\frac{\tan d}{\rho} = \frac{\tan d_s}{r} \text{ or } \rho = \frac{r}{\tan d_s}\tan d$$

then $$d\rho = \frac{r}{\tan d_s} d(\tan d)$$

Substituting in the equation for dM, the values of ρ and $d\rho$ just derived in terms of tan d and d(tan d), we have the general differential equation:

$$dM = 2\pi q \left(\frac{r}{\tan d_s}\right)^2 (\tan d)^2 \frac{r}{\tan d_s} d(\tan d).$$

or $$dM = 2\pi \left(\frac{r}{\tan d_s}\right)^3 q\ (\tan d)^2 d(\tan d).$$

putting $\psi(\tan d)$ for q

$$dM = 2\pi \left(\frac{r}{\tan d_s}\right)^3 \psi(\tan d)\cdot(\tan d)^2\cdot d(\tan d)$$

Integrating,

$$M = 2\pi \left(\frac{r}{\tan d_s}\right)^3 \int_0^{\tan d_s} \psi(\tan d)\cdot(\tan d)^2\cdot d(\tan d).$$

Now, q_{ns}, the nominal stress intensity in the outer surface by the elastic formula, is $\frac{Mr}{J} = \frac{2M}{\pi r^3}$

hence, $$q_{ns} = \left(\frac{2}{\pi r^3}\right) 2\pi \left(\frac{r}{\tan d_s}\right)^3 \int_0^{\tan d_s} \psi(\tan d)\cdot(\tan d)^2\cdot d(\tan d)$$
$$= \frac{4}{\tan^3 d} \cdot \int_0^{\tan d_s} \psi(\tan d)\cdot\tan^2 d\cdot d(\tan d)$$

Put U for $\frac{4}{\tan^3 d_s}$ and V for the $\int_0^{\tan d_s} \psi(\tan d)\cdot\tan^2 d\cdot d(\tan d)$.

Then $q_{ns} = U\,V$. Differentiate this with regard to tan d_s, to get the slope of the curve of q_{ns} versus tan d_s, at the abcissa tan d_s. This curve is the one determined experimentally and known from test.

$$\frac{d}{d(\tan d_s)}(q_{ns}) = U\frac{d}{d(\tan d_s)}(V) + V\frac{d}{d(\tan d_s)}(U)$$
$$= \frac{4}{\tan^3 d_s}\frac{d}{d(\tan d_s)}\left(\int_0^{\tan d_s} \psi(\tan d)\tan^2 d\ d(\tan d)\right)$$
$$+ \left(\int_0^{\tan d_s} \psi(\tan d)\ \tan^2 d\ d(\tan d)\right)\frac{d}{d(\tan d_s)}\left(\frac{4}{\tan^3 d_s}\right)$$

In the first term of this expression, the value of the

$$\frac{d}{d(\tan d_s)}\left(\int_0^{\tan d_s} \psi(\tan d)\ \tan^2 d\ d(\tan d)\right)$$

is the differential of an integral with regard to its upper limit and has the value $\psi(\tan d_s)\ \tan^2 d_s$. For this to be true it is necessary that $\psi(\tan d)$ be a continuous finite singled-valued function of tan d. We know that q is such a function of tan d. In the second term of the differential of q_{ns} the value of

$$V = \left(\int_0^{\tan d_s} \psi(\tan d)\tan^2 d \; d(\tan d)\right) \text{ is } q_{ns}\frac{\tan^3 d_s}{4}$$

Making these substitutions, we have

$$\frac{d}{d(\tan d_s)}(q_{ns}) = \frac{4}{\tan^3 d_s}(\psi(\tan d_s)\tan^2 d_s) + \frac{q_{ns}\tan^3 d_s}{4}\left(\frac{4(-3)}{\tan^4 d_s}\right)$$
$$= \frac{4\,\psi(\tan d_s)}{\tan d_s} - \frac{3\,q_{ns}}{\tan d_s}$$

But $\psi(\tan d_s)$ is the real value of q_s. Hence

$$\frac{d}{d(\tan d_s)}(q_{ns}) = \frac{4q_s}{\tan d_s} - \frac{3q_{ns}}{\tan d_s}$$

Multiplying through by tan d, we obtain

$$\tan d_s \frac{d}{d(\tan d_s)}(q_{ns}) = 4q_s - 3q_{ns}$$

Change signs and add q to each side.

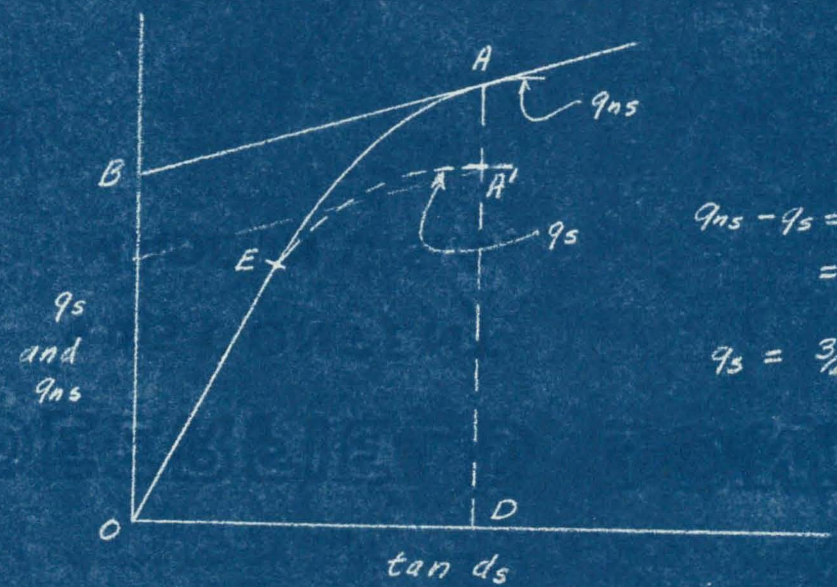
Then $$\left(q_{ns} - \tan d_s \frac{d}{d(\tan d_s)}(q_{ns})\right) = 4(q_{ns} - q_s)$$

The left hand side of the last equation is the intercept on the q-axis of the tangent to the curve of which the abcissa is tan d. The difference between the nominal and the true shearing stresses will then be one-quarter of that intercept. It is then apparent that if the curve of q versus tan d is ~~tangent~~ horizontal at failure, the true shearing stress will be three-fourths of the nominal shearing stress.

Upton states that a similar proof may be demonstrated for rectangular beams in flexure. In this case however, the correction factor is two-thirds. The nominal stress would be that computed by the formula, $f = \frac{Mc}{I}$, and the true stress at failure, providing the nominal stress-strain curve was horizontal at failure, would be $f = \frac{2}{3}\frac{Mc}{I}$.

Nominal Stress-Strain Diagrams

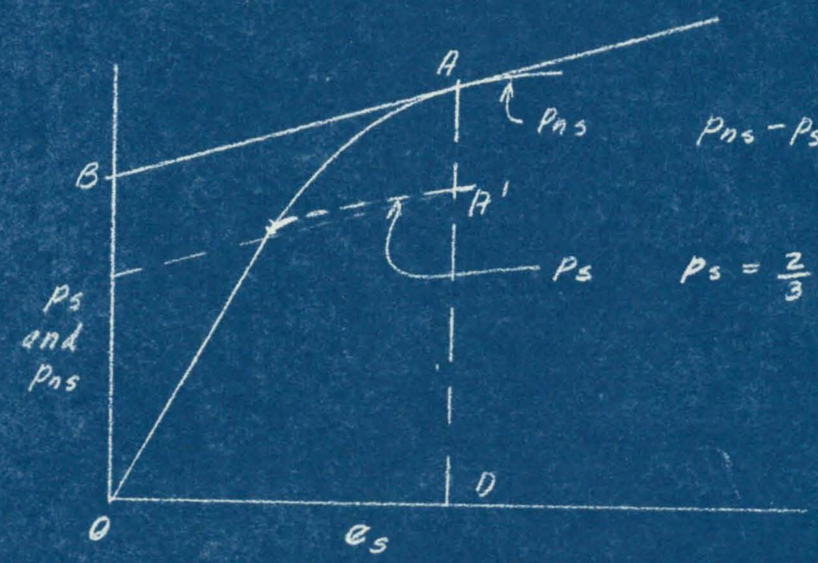
Showing Relation of True to Nominal Stresses



$$q_{ns} - q_s = AD - A'D = \frac{OB}{4}$$

$$q_s = \tfrac{3}{4}\, q_{ns} = \tfrac{3}{4}\, \frac{Mr}{J}$$

Circular Shaft in Torsion.



$$p_{ns} - p_s = AD - A'D = \frac{OB}{3}$$

$$p_s = \frac{2}{3}\, p_{ns} = \frac{2}{3}\, \frac{Mc}{I}$$

Rectangular Beam in Flexure

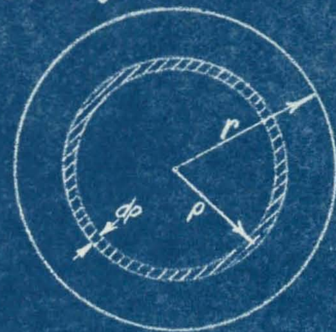
APPENDIX II

DERIVATION OF FORMULAE GIVEN IN FIGS.1,2,&3

1. Cylindrical Member in Torsion.

Elastic Theory (derivation in any mechanics text).

$$V_m = \frac{Mr}{J}$$



where

V_m = max. shearing stress.
M = max. twisting moment.
r = radius
J = polar Moment of Inertia
$= \frac{\pi d^4}{32}$

Rectangular Stress Distribution (Fig.1(c)).

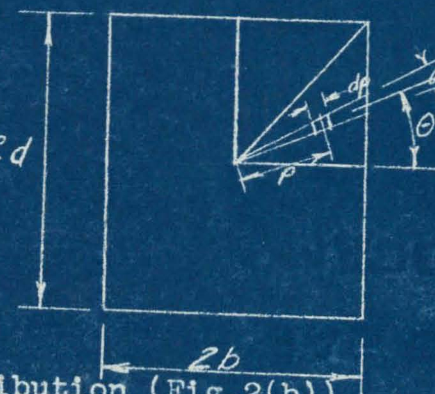
$$dM = 2\pi\rho \cdot d\rho \cdot V_m \cdot \rho$$
$$M = 2\pi V_m \int_0^r \rho^2 d\rho$$
$$= 2\pi V_m \cdot \frac{r^3}{3}$$
$$= \tfrac{2}{3}\pi r^3 \cdot V_m$$
$$V_m = \tfrac{3}{2}\,\frac{M}{\pi r^3} = \tfrac{3}{4}\,\frac{Mr}{J}$$

2nd Degree Parabola Stress Distribution.

$$dM = 2\pi\rho \cdot d\rho \left[1 - \frac{(r-\rho)^2}{r^2}\right] V_m \cdot \rho$$
$$M = 2\pi V_m \int_0^r \left(\rho^2 - \rho^2 + \frac{2\rho^3}{r} - \frac{\rho^4}{r^2}\right) d\rho$$
$$= 2\pi V_m \left(\frac{r^3}{2} - \frac{r^3}{5}\right)$$
$$= \tfrac{3}{5}\pi r^3 \cdot V_m$$
$$V_m = \tfrac{5}{3}\,\frac{M}{\pi r^3} = \tfrac{5}{6}\,\frac{Mr}{J}$$

2. Rectangular Member in Torsion.

Elastic Theory (Bach)



$$V_m = K\frac{M}{8b^2d}$$

where $K = \left(3 + \frac{2.6}{0.45 + d/b}\right)$.

for $\frac{d}{b} = 1$, $V_m = 4.80\frac{M}{8b^2d} = 0.60\,\frac{M}{b^2d}$

for $\frac{d}{b} = \frac{3}{2}$, $V_m = 4.33\,\frac{M}{8b^2d} = 0.542\,\frac{M}{b^2d}$

for $\frac{d}{b} = 2$, $V_m = 4.06\frac{M}{8b^2d} = 0.517\,\frac{M}{b^2d}$

Rectangular Stress Distribution (Fig.2(b)).

$$dM = \rho\cdot d\rho\cdot d\theta\cdot V_m\cdot\rho$$

$$M = 4V_m\int_0^{\tan^{-1}\frac{d}{b}}\int_0^{b\cdot\sec\theta}\rho^2\,d\theta\,d\rho + 4V_m\int_0^{\tan^{-1}\frac{b}{d}}\int_0^{d\cdot\sec\theta}\rho^2\,d\theta\,d\rho$$

$$= 4V_m\int_0^{\tan^{-1}\frac{d}{b}}\left|\frac{\rho^3}{3}\right|_0^{b\cdot\sec\theta}d\theta + 4V_m\int_0^{\tan^{-1}\frac{b}{d}}\left|\frac{\rho^3}{3}\right|_0^{d\cdot\sec\theta}d\theta$$

$$= \frac{4}{3}b^3V_m\int_0^{\tan^{-1}\frac{d}{b}}\sec^3\theta\,d\theta + \frac{4}{3}d^3V_m\int_0^{\tan^{-1}\frac{b}{d}}\sec^3\theta\,d\theta$$

$$= \frac{4}{3}b^3V_m\left|\frac{\sin\theta}{2\cos^2\theta} + \frac{1}{2}\int\sec\theta\,d\theta\right|_0^{\tan^{-1}\frac{d}{b}}$$

$$+ \frac{4}{3}d^3V_m\left|\frac{\sin\theta}{2\cos^2\theta} + \frac{1}{2}\int\sec\theta\,d\theta\right|_0^{\tan^{-1}\frac{b}{d}}$$

$$M = \frac{4}{3}b^3V_m\left|\frac{\sin\theta}{2\cos^2\theta} + \frac{1}{4}\log\frac{1+\sin\theta}{1-\sin\theta}\right|_0^{\tan^{-1}\frac{d}{b}}$$

$$+ \frac{4}{3}d^3V_m\left|\frac{\sin\theta}{2\cos^2\theta} + \frac{1}{4}\log\frac{1+\sin\theta}{1-\sin\theta}\right|_0^{\tan^{-1}\frac{b}{d}}$$

for $\frac{d}{b} = 1$, $V_m = \frac{M}{2.4\,b^3} = 0.416\,\frac{M}{b^3}$, or $V_m = 0.695\,K\frac{M}{8b^3}$

for $\frac{d}{b} = 3/2$, — — — — — — — — — $V_m = 0.606\,K\frac{M}{8b^2d}$

for $\frac{d}{b} = 2$, — — — — — — — — $V_m = 0.527\,K\frac{M}{8b^2d}$

3. Rectangular Member in Flexure.

Elastic Theory (derivation in any mechanics text).

$$M = \frac{fI}{c},$$
$$f = \frac{Mc}{I}$$

Rectangular Stress Distribution(Fig.3(c)).

$$M = f \cdot b \cdot \frac{d}{2} \cdot \frac{d}{2}$$
$$f = 4\frac{M}{bd^2} = \frac{2}{3}\frac{Mc}{I}$$

2nd Degree Parabola Stress Distribution on Tension Side, (Fig.3(d)).

$$M = \frac{2}{3} \cdot f_t \cdot b \cdot \frac{d}{2}\left(\frac{5}{16} + \frac{1}{3}\right)d$$
$$= \frac{31}{144} f_t \cdot bd^2$$
$$f_t = \frac{144}{31}\frac{M}{bd^2} = \frac{144}{186}\frac{Mc}{I} = 0.775\frac{Mc}{I}$$

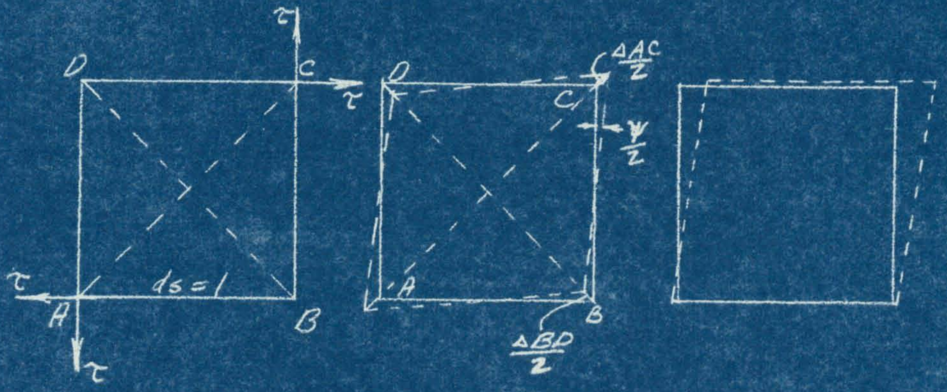
Rectangular Stress Distribution on Tension Side, (Fig.3(e)).

$$M = f_t \cdot b \cdot \frac{d}{2}\left(\frac{d}{4} + \frac{d}{3}\right)$$
$$= \frac{7}{24} f_t \cdot bd^2$$
$$f_t = \frac{24}{7}\frac{M}{bd^2} = \frac{24}{42}\frac{Mc}{I} = 0.572\frac{Mc}{I}$$

APPENDIX III

ANALYSIS OF CIRCULAR TORSION SPECIMEN BY MORSCH

Consider a square element ABCD parallel to the surface of the cylinder. On each edge there will be a shear τ which will be a unit stress if we call the dimensions of the square unity and consider that the forces acting in the plane of the square act over a unit of depth as well. Under the action of the shearing forces τ, direct stresses will be set up along the two diagonals, a tension along AC and a compression along BD, each equal in magnitude to τ.



If we call the modulus of elasticity and Poisson's ratio for tension E_t and $1/m_t$, respectively, and for compression E_c and $1/m_c$, the following relations are true.

$$\frac{\Delta AC}{2} = \frac{1}{2}\sqrt{2}\left(\frac{\tau}{E_t} + \frac{1}{m_c}\cdot\frac{\tau}{E_c}\right)$$

$$\frac{\Delta BD}{2} = \frac{1}{2}\sqrt{2}\left(\frac{\tau}{E_c} + \frac{1}{m_t}\cdot\frac{\tau}{E_t}\right)$$

Half the shearing strain on the element will be

$$\frac{\psi}{2} = \left\{\frac{\Delta AC}{2} + \frac{\Delta BD}{2}\right\}\frac{1}{2}$$

$$\frac{\psi}{2} = \frac{1}{2}\sqrt{2}\cdot\tau\cdot\left(\frac{1}{E_t} + \frac{1}{m_c E_e} + \frac{1}{E_c} + \frac{1}{m_t E_t}\right)\frac{1}{\sqrt{2}}$$

$$\psi = \tau\left\{\frac{1}{E_t}\left(1 + \frac{1}{m_t}\right) + \frac{1}{E_c}\left(1 + \frac{1}{m_c}\right)\right\}$$

The longitudinal strain along AB will be

$$\varepsilon = \left(\frac{\Delta AC}{2} - \frac{\Delta BD}{2}\right)\frac{1}{\sqrt{2}}$$

$$\varepsilon = \frac{1}{2}\sqrt{2}\,\tau\left(\frac{1}{E_t} + \frac{1}{m_c E_c} - \frac{1}{E_c} - \frac{1}{m_t E_t}\right)\frac{1}{\sqrt{2}}$$

$$\varepsilon = \frac{\tau}{2}\left\{\frac{1}{E_t}\left(1 - \frac{1}{m_t}\right) - \frac{1}{E_c}\left(1 - \frac{1}{m_c}\right)\right\}$$

For use in these formulae Morsch used values of Poisson's ratio obtained from other tests, which were

$$1/m_t = 1/10 , \quad 1/m_c = 1/5$$

Since the direct stress on the diagonal equals τ, Morsch set up the following table using experimental values for E_t, E_c.

σ or τ	E	E	$10^6\psi$	$10^6\varepsilon$
kg./cm.^{2}	kg./cm.^{2}	kg./cm.^{2}		
2	316 000	270 000	15.74	0.80
4	310 000	257 000	32.60	1.84
6	304 000	240 000	51.18	3.36
8	298 000	218 000	72.56	5.76
10	293 000	193 000	98.00	9.65
11	290 500	175 000	114.73	13.15

From the testing, the ultimate strain $10^6\psi$ was found to be 196. Using this value and the table above, the shearing stress distribution drawn in the accompanying figure, extrapolating for the last portion of the curve. The resisting moment of the cylinder will be

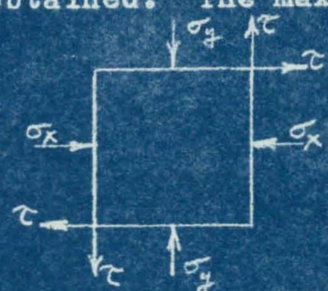
$$M = 2\pi \; \tau \, dx \, x^2$$

the integral being equal to the moment of inertia of the semi-parabolic area representing the shearing stresses. By determining the value of the integral graphically, the resisting moment was found to be 220,000 cm.kg., which agrees very well with the observed value of 233,000 cm.kg.

Next, the values of ε were laid off downward from the baseline OB in the figure, it being again necessary to extrapolate for the last portion of the curve. Since the outer portion of the cylinder has much greater values of ε than the inner, it will tend to extend much more, putting the inner portion into tension, and therefore the outer portion into compression to maintain equilibrium. To balance these opposing forces, Morsch assumed a constant value of E for the longitudinal stresses, which are all quite small, and by trial and error established a new base line GH such that the total longitudinal tension equalled the total longitudinal compression. The resulting compression at the surface, assuming E to be 310,000 kg./cm.^{2}, became

$$\sigma = \frac{15\ 310000}{10^6} = 4.6 \text{ kg./cm}^2$$

Morsch then goes on to explain that there are circumferential stresses also set up which will be approximately equal to the longitudinal stresses just obtained. The maximum principal stresses will then be



$$\sigma_1 = -\frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\ (\sigma_x - \sigma_y)^2 + 4\tau^2$$

or since $\sigma_x = \sigma_y = 4.6$ and $\tau = 16$ kg./cm.^{2}

$$\sigma_1 = 16 - 4.6 = 11.4 \text{ kg./cm.}^2$$

which gives excellent agreement with the direct tension ultimate strength of 11.5 kg./cm.^{2}. Due to the symmetrical nature of the forces acting upon the differential element, the principal stresses must necessarily act at 45° to the horizontal, as in actual test specimens.

APPENDIX III